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Algorithms for the polynomial zeros of degree 2 of the 3-*j* and the 6-*j* coefficients

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Abstract. Simple algorithms, which are based on the principle of factorisation of an integer, are proposed to generate the polynomial zeros of degree 2 of the 3-*j* and 6-*j* coefficients.

1. Introduction

In recent years considerable interest has been generated by the paper of Koozekanani and Biedenharn (1974) (see also Biedenharn and Louck 1981) to study non-trivial or polynomial zeros of the 3-*j* and 6-*j* angular momentum coefficients. Systematic treatment of the study of the polynomial zeros of degree 1 of the 6-*j* coefficient has been made from the point of view of the realisations of the algebra of exceptional Lie groups by Van den Berghe *et al* (1983, 1984) or from the point of view of multiplicative Diophantine equations by Brudno (1985), Brudno and Louck (1985), Srinivasa Rao and Rajeswari (1984, 1985a, b, 1986) and Srinivasa Rao *et al* (1988). More recently, Louck and Stein (1987) and Beyer *et al* (1986) have shown that solutions of the quadratic Diophantine equation, known as Pell's equation (see Dickson 1952), are related to polynomial zeros of degree 2 of the 3-*j* and 6-*j* coefficients. They showed that this relation involves transformations of quadratic forms over the integers and the orbit classification of zeros of Pell's equation. They gave an algorithm for determining numerically the fundamental solutions of Pell's equation. However, they emphasise that, in the case of the 6-*j* coefficient, the zeros obtained by them do not include *all* the polynomial zeros of degree 2.

In § 2 we give the key definitions and in § 3 we discuss our algorithms for generating the polynomial zeros of degree 2 of the 3-*j* and 6-*j* coefficients, using the principle of factorisation of an integer and the solution to a quadratic (in the case of the 3-*j*) or a cubic (in the case of the 6-*j*) equation. In § 4 we present a discussion of the results based on the algorithms.

2. Key definitions

It has been shown by one of us (Srinivasa Rao 1978) that the 3-*j* coefficient can be

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represented by a set of six ${}_3F_2(1)$ factors:

$$
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}
= \delta(m_1+m_2+m_3, 0)(-1)^{\sigma(pqr)} \prod_{i,k=1}^{3} [R_{ik}!/(J+1)!]^{1/2}
\times [\Gamma(1-A, 1-B, 1-C, D, E)]^{-1} {}_3F_2(A, B, C; D, E; 1) \tag{1}
$$

where

$$
\begin{aligned}
&A = -R_{2p} \qquad B = -R_{3q} \qquad C = -R_{1r} \qquad D = R_{3r} - R_{2p} + 1 \\
&E = R_{2r} - R_{3q} + 1 \\
&\Gamma(x, y, \ldots) = \Gamma(x)\Gamma(y)\ldots \qquad J = j_1 + j_2 + j_3
\end{aligned} \tag{2}
$$

and

$$
\sigma(pqr) = \begin{cases} R_{3p} - R_{2q} & \text{for even permutations} \\ R_{3p} - R_{2q} + J & \text{for odd permutations} \end{cases}
$$

for all permutations of $(pqr) = (123)$, with R_{ik} being elements of the 3×3 Regge (1959) array:

$$
\left\| \begin{matrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{matrix} \right\|. \tag{3}
$$

By setting any one of the numerator parameters, say C, to $-n$ and equating the sum of the $(n+1)$ terms to zero, one obtains the constraint equation which must be satisfied by the numerator and denominator parameters of the ${}_3F_2(1)$ for realising the polynomial zeros of degree n. Polynomial zeros of degree 2 are obtained when we set $C = -2$ and subject the parameters A, B, D and E to the constraint equation:

$$
A(A+1)B(B+1) - 2AB(D+1)(E+1) + D(D+1)E(E+1) = 0 \tag{4}
$$

which has been related by Louck and Stein (1987) to the Pell equation.

The 6-j coefficient can be represented by a set I of three ${}_4F_3(1)$ factors (Srinivasa Rao *et al* 1975) as

$$
\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{E+1} N\Gamma(1-E)
\times [\Gamma(1-A, 1-B, 1-C, 1-D, F, G)]^{-1} {}_4F_3(A, B, C, D; E, F, G; 1) \tag{5}
$$

where

$$
\begin{aligned}
&N = \Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf) \\
&A = -R_{1p} \qquad B = -R_{2p} \qquad C = -R_{3p} \qquad D = -R_{4p} \\
&E = -R_{1p} - R_{2p} - R_{3q} - R_{4r} - 1 \qquad F = R_{3q} - R_{3p} + 1 \qquad G = R_{4r} - R_{4p} + 1
\end{aligned} \tag{6}
$$

for $(pqr) = (123)$ cyclic; with R_{ik} being the elements of the Bargmann (1962) and Shelepin (1964) array:

$$
\left\| \begin{matrix} \beta_1-\alpha_1 & \beta_2-\alpha_1 & \beta_3-\alpha_1 \\ \beta_1-\alpha_2 & \beta_2-\alpha_2 & \beta_3-\alpha_2 \\ \beta_1-\alpha_3 & \beta_2-\alpha_3 & \beta_3-\alpha_3 \\ \beta_1-\alpha_4 & \beta_2-\alpha_4 & \beta_3-\alpha_4 \end{matrix} \right\| \tag{7}
$$

where

$$\alpha_1 = a+b+e \qquad \alpha_2 = c+d+e \qquad \alpha_3 = a+c+f \qquad \alpha_4 = b+d+f$$

$$\beta_1 = a+b+c+d \qquad \beta_2 = a+d+e+f \qquad \beta_3 = b+c+e+f$$

and

$$\Delta(xyz) = [(-x+y+z)!(x-y+z)!(x+y-z)!/(x+y+z+1)!]^{1/2}.$$

Equivalently, the 6-*j* coefficient can also be represented by a set II of four ${}_4F_3(1)$ factors (Srinivasa Rao and Venkatesh 1977) as

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{A'-2} N\Gamma(A')[\Gamma(1-B', 1-C', 1-D', E', F', G')]^{-1}$$
$$\times {}_4F_3(A', B', C', D'; E', F', G'; 1) \qquad (8)$$

where, using the conditions satisfied by the R_{ik}, the numerator and denominator parameters of this set can be shown to be

$$\begin{aligned} &A' = R_{q2} + R_{r1} + R_{s3} + 2 \qquad B' = -R_{p1} \qquad C' = -R_{p2} \qquad D' = -R_{p3} \\ &E' = R_{q1} - R_{p1} + 1 \qquad F' = R_{r1} - R_{p1} + 1 \qquad G' = R_{s1} - R_{p1} + 1 \end{aligned} \qquad (9)$$

for $(pqrs) = (1234)$ cyclically. Obviously, polynomial zeros of degree n arise when the sum of the first $n+1$ terms of the ${}_4F_3(1)$ occurring in (5) or (8) adds to zero. The polynomial zeros of degree 2, studied by Beyer *et al* (1986), are obtained when $D = -2$ and the parameters satisfy the condition:

$$A(A+1)B(B+1)C(C+1) - 2ABC(E+1)(F+1)(G+1)$$
$$+ E(E+1)F(F+1)G(G+1) = 0 \qquad (10)$$

or

$$A'(A'+1)B'(B'+1)C'(C'+1) - 2A'B'C'(E'+1)(F'+1)(G'+1)$$
$$+ E'(E'+1)F'(F'+1)G'(G'+1) = 0. \qquad (10')$$

In the following section we discuss simple algorithms to generate *all* the polynomial zeros of degree 2 of the 3-*j* and 6-*j* coefficients.

3. Algorithms to generate degree 2 polynomial zeros

Recently, Louck and Stein (1987) and Beyer *et al* (1986) studied the polynomial zeros of degree 2 of the 3-*j* and 6-*j* coefficients in those special cases where the three-term general forms (4), (10) and (10′) can be factored into two parts, one of which is a quadratic form related to the generalised Pell equation:

$$x^2 - \xi y^2 = \eta \qquad (11)$$

where ξ is a positive integer and η is a positive or negative integer. (The other part has no non-trivial zeros.) The authors then exploit the known orbit classification of solutions of Pell's equation to obtain classes of polynomial zeros of degree 2 of the 3-*j* and 6-*j* coefficients. However, they emphasise that their method does not generate *all* the polynomial zeros of degree 2.

In the case of the 3-*j* coefficient, we consider D and E, the denominator parameters of the set of ${}_3F_2(1)$ factors, to take integer values with $D \leqslant E$ ($E = 1, 2, \ldots, N$). Let $c_1 = (D+1)(E+1)$ and $c_2 = DEc_1$. Then (4) becomes

$$AB(A+1)(B+1) - 2c_1AB = -c_2 \qquad (12)$$

which can be simplified into

$$u(u+v)=-c_2 \tag{13}$$

where $u=AB$ and $v=A+B-2c_1+1$. Since c_2 is an integer, we find all the divisors of c_2. If u is a divisor, then by (13), $v=-c_2/u-u$. Given $AB=u$ and $A+B=v+2c_1-1$, using the elementary identity: $(A-B)^2=(A+B)^2-4AB$, $A-B$ is found. Since we are solving (4) for polynomial zeros of degree 2, $C=-2$, A and B being negative integer parameters, they must each be less than or equal to -2 so that we must have $AB\geqslant 4$ and $A+B\leqslant -4$. From $A+B$ and $A-B$, we solve (algebraically) for A and B. Thus, we get the set of values of A and B (for a given D and E) from all the divisors of c_2. We now state our algorithm for the generation of polynomial zeros of degree 2 of the 3-j coefficient.

Algorithm 1.

(i) Choose E to take values 1 to N and D to take values $1\leqslant D\leqslant E$ by arranging these into a nest of loops.

(ii) Find the divisors of $c_2\equiv c_1(DE)$, where $c_1\equiv(D+1)(E+1)$.

(iii) For each divisor u, find $v=-c_2/u-u$ and hence $A+B-2c_1+1$.

(iv) For $u\geqslant 4$ and $A+B\leqslant -4$, let $x=(A+B)^2-4u$. Then if $x<0$, no solution exists and so go to (iii); if $x=0$, $A=B=(v+2c_1-1)/2$ is a solution; if $x>0$, find m such that $x=m^2$, for such an m, $A=(v+2c_1-1+m)/2$ and $B=(v+2c_1-1-m)/2$ is a solution, provided $A\geqslant -2$ and $B\geqslant -2$.

(v) Having found the solution (A and B for a given D and E, when $C=-2$), repeat the procedure for the next divisor of c_2, i.e. go to step (iii), to find the set of all A, B which satisfy (12).

(vi) For each A, B, D, E (and $C=-2$), find the polynomial zero of degree 2 of the 3-j coefficient.

In the case of the 6-j coefficient, the equation to be solved is (10) or (10′). We choose to work with set II of ${}_4F_3(1)$, since the denominator parameters E', F' and G' are required to be positive integers, let them take integer values 1 to N, with $E'\geqslant F'\geqslant G'$. Let $c_1'=(E'+1)(F'+1)(G'+1)$, $c_2'=c_1'E'F'G'$, $c_3=E'+F'+G'+2$ and $c_4=2c_1'-c_3$. Then (10′) becomes

$$u'(u'+v')=-c_2' \tag{14}$$

where $u'=A'B'C'$ and $v'=A'B'+(A'+B')C'+c_4$ using the Saalschutzian condition. As before, since c_2' is an integer, we find all the divisors of c_2', say d_i. Since $u'=A'B'C'$, we need to find the divisors of u' and let these be e_i. If we identify C' as one of the divisors of u' then $A'B'=d_i/e_i$ and from the definition of v', we have $A'+B'=(v'-d_i/e_i-c_4)/e_i$. Knowing $A'B'$ and $A'+B'$, as in the case of the 3-j coefficient, we can find A' and B'. Thus, in the case of the 6-j coefficient, we have to find the divisors twice. The algorithm for the generation of polynomial zeros of degree 2 of the 6-j coefficient, using the method of divisors can now be stated as follows:

Algorithm 2.

(i) Choose E' to take values 1 to N, and let the values of F' and G' be: $1\leqslant F'\leqslant E'$ and $1\leqslant G'\leqslant F'$. E', F' and G' are arranged into a nest of loops.

(ii) Find the values of $c_1'=(E'+1)(F'+1)(G'+1)$, $c_2'=c_1'E'F'G'$, $c_3=E'+F'+G'+2$ and $c_4=c_3-2c_2'$.

(iii) Find the divisors of c_2'. Let these be d_i. If $u'=d_i$, then from (18), $v'=-c_2/d_i-d_i$.

(iv) Find the divisors of u'. Let these be e_i. If $C' = e_i$, then $A'B' = (u'/C' =)d_i/e_i$ and $A' + B' = (v' - A'B' - c_4)/e_i$.

(v) For $u' \geqslant 8$ and $A' + B' + C' \leqslant -6$, let $x' = (A' + B')^2 - 4u'$. Then if $x' < 0$, no solution exists and so go to (iv); if $x' = 0$, $A' = B' = (v' - A'B' - c_4)/2e_i$; if $x' > 0$, find n such that $x' = n^2$ for such an n, $A' = (v' - A'B' - c_4 + ne_i)/2e_i$ and $B' = (v' - A'B' - c_4 - ne_i)/2$, is a solution provided $A' \leqslant -2$ and $B' \leqslant -2$.

(vi) Having found the solution (A' and B' for a given divisor of u', viz $C' = e_i$, when $D' = -2$) repeat the procedure for the next divisor of C', i.e. go to step (iv).

(vii) After completing the search for all the divisors of u', (namely for all e_i), go to step (iii) for the next divisor of c_2, to find the set of all A', B' and C' which satisfy (14).

(viii) For each A', B', C', E', F', G' and $D' = -2$, find the polynomial zero of degree 2 of the 6-j coefficient.

In algorithm 2, there arose the necessity of finding the divisors of c_2' first, and later of finding the divisors of u'. Since the number of divisors is large for large integers, we propose an alternate algorithm to generate the polynomial zeros of degree 2 of the 6-j coefficient. To this end, after obtaining the solution for (18) by finding the divisors of c_2', using the Saalschitzian condition, it is straightforward to show that the expression for v' can be rewritten as a cubic equation for C',

$$C'^3 + a_2 C'^2 + a_1 C' + a_0 = 0 \tag{15}$$

where

$$a_2 = (c_3 + 1) \qquad a_1 = v - c_4 = -(c_2/d_i + d_i + c_4) \qquad a_0 = -d_i.$$

To get the solutions of this cubic equation, let

$$q = \tfrac{1}{2}a_1 - \tfrac{1}{9}a_2^2 \qquad r = \tfrac{1}{6}(a_1 a_2 - 3a_0) - \tfrac{1}{27}a_2^3.$$

It is well known (Abramowitz and Segun 1968) that, if $q^3 + r^2 > 0$, (15) has one real root and a pair of complex roots; if $q^3 + r^2 = 0$, all the roots are real and at least two are equal; and if $q^3 + r^2 < 0$ all the roots are real. The roots of (15) can be written as

$$\begin{aligned}
z_1 &= (s_1 + s_2) - \tfrac{1}{3}a_2 = \alpha^{1/3} + \beta^{1/3} - \tfrac{1}{3}a_2 \\
z_2 &= -\tfrac{1}{2}(s_1 + s_2) - \tfrac{1}{3}a_2 + \tfrac{1}{2}\mathrm{i}\sqrt{3}(s_1 - s_2) \\
&= [\alpha \exp(2\pi\mathrm{i})]^{1/3} + [\beta \exp(-2\pi\mathrm{i})]^{1/3} - \tfrac{1}{3}a_2 \\
z_3 &= -\tfrac{1}{2}(s_1 + s_2) - \tfrac{1}{3}a_2 - \tfrac{1}{2}\mathrm{i}\sqrt{3}(s_1 - s_2) \\
&= [\alpha \exp(-2\pi\mathrm{i})]^{1/3} + [\beta \exp(2\pi\mathrm{i})]^{1/3} - \tfrac{1}{3}a_2
\end{aligned}$$

where $\alpha = r + (q^3 + r^2)^{1/2}$ and $\beta = r - (q^3 + r^2)^{1/2}$. The roots z_1, z_2, z_3 of the cubic equation (15) satisfy the relations:

$$\begin{aligned}
&z_1 z_2 z_3 = -a_0 \\
&z_1 z_2 + z_1 z_3 + z_2 z_3 = a_1 \\
&z_1 + z_2 + z_3 = -a_2.
\end{aligned} \tag{16}$$

It is to be noted that since we are interested in only integer solutions of (19), the roots of (20) must all be real.

We now propose the alternate algorithm for generating the polynomial zeros of degree 2 of the 6-j coefficient.

Algorithm 2a.

(i) Choose E' to take the values 1 to N, and let the values of F' and G' be: $1 \leqslant F' \leqslant E'$ and $1 \leqslant G' \leqslant F'$. E', F' and G' are arranged into a nest of loops.

(ii) Find the values of

$$c_1' = (E'+1)(F'+1)(G'+1) \qquad c_2' = c_1'E'F'G'$$

$$c_3 = E'+F'+G' \qquad c_4 = c_3 - 2c_1' + 2.$$

(iii) Find the divisors of c_2'. Let these be d_i. If $u' = d_i$, then from (14), $v' = -c_2/d_i - d_i$.

(iv) Let

$$q = \tfrac{1}{3}a_1 - \tfrac{1}{9}a_2^2 \qquad r = \tfrac{1}{6}(a_1a_2 - 3a_0) - \tfrac{1}{27}a_2^3 \qquad D = q^3 + r^2$$

where $a_0 = -d_i$, $a_1 = -(c_3+1)$ and $a_2 = -c_2/d_i - d_i$. Also, let $\alpha = r + \sqrt{D}$ and $\beta = r - \sqrt{D}$.

(v) If $D > 0$, then $\alpha \neq \beta$, and hence only one real root exists:

$$z = \begin{cases} \alpha^{1/3} + \beta^{1/3} - \tfrac{1}{3}a_2 & \text{when } \alpha \geqslant 0,\ \beta \geqslant 0 \\ |\alpha|^{1/3} - \tfrac{1}{3}a_2 & \text{at } \alpha = \beta, \text{ when } \alpha < 0,\ \beta > 0 \\ -\tfrac{1}{3}a_2 & \text{at } \alpha = -\beta, \text{ when } \alpha < 0,\ \beta < 0 \text{ or } \alpha < 0,\ \beta > 0. \end{cases} \tag{17}$$

(vi) If $D = 0$, then $\alpha = \beta = r$, and hence the real roots are

$$z_1 = 2r^{1/3} - \tfrac{1}{3}a_2$$

$$z_2 = z_3 = -r^{1/3} - \tfrac{1}{3}a_2 \qquad \text{when } \alpha < 0,\ \beta < 0.$$

(vii) If $D < 0$, $s_1 = [\beta \exp(\mathrm{i}\theta)]^{1/3}$ and $s_2 = [\beta \exp(-\mathrm{i}\theta)]^{1/3}$ where $\sin\theta = \alpha/\beta$ and $\cos\theta = r/\beta$, then the real roots are

$$z_1 = 2\beta^{1/3}\cos(\tfrac{1}{3}\theta) - \tfrac{1}{3}a_2$$

$$z_2 = -2\beta^{1/3}\cos\left(\frac{\pi-\theta}{3}\right) - \tfrac{1}{3}a_2$$

$$z_3 = -2\beta^{1/3}\cos\left(\frac{\pi+\theta}{3}\right) - \tfrac{1}{3}a_2.$$

(viii) Having found the real roots of C' from the cubic equation (19) satisfied by it, from (v), (vi) or (vii), if $C' = e_i$, then

$$A'B' = u'/C' = d_i/e_i \qquad A'+B' = (v' - A'B' - c_4)/e_i. \tag{18}$$

(ix) For $u' \geqslant 8$ and $A'+B'+C' \leqslant -6$, let $\xi' = (A'+B')^2 - 4u'$. Then if $\xi' < 0$, no solution exists and so go to (viii); if $\xi' = 0$, $A' = B' = (v' - d_i/e_i - c_4)/2e_i$; if $\xi' > 0$, find n such that $\xi' = n^2$ for such an n, $A' = (v' - d_i/e_i - c_4 + ne_i)/2$ and $B' = (v' - d_i/e_i - c_4 - ne_i)/2$, is a solution provided $A' \leqslant -2$ and $B' \leqslant -2$.

(x) Having found the solution (A' and B' for a given root $C' = e_i$, when $D' = -2$) repeat the procedure for the other roots of the cubic equation, if any, by going back to step (viii).

(xi) Go to step (iii) for the next divisor of c_2',to find all the A', B', C' which satisfy (14).

(xii) For each A', B', C', E', F', G' and $D' = -2$, find the polynomial zeros of degree 2 of the 6-j coefficient.

4. Results and discussion

Algorithm 1 and algorithms 2 and 2a for generating the polynomial zeros of degree 2 of the 3-j and 6-j coefficients, respectively, have been used to produce tables 1 and 2, for $J=j_1+j_2+j_3 \leqslant 143$ in the case of the 3-j coefficient and a, b, c, d, e or $f \leqslant 14$ (Srinivasa Rao and Rajeswari 1985a, table IV) in the case of the 6-j coefficient. Both Fortran and common Lisp were used to write the programs based on algorithm 1, and

Table 1. Polynomial zeros of degree 2 of the 3-j coefficient. In the last column the value of $J=j_1+j_2+j_3$ is given. The zeros listed here are inequivalent polynomial zeros of degree 2.

j_1	j_2	j_3	m_1	m_2	m_3	J
6	4	4	2	0	-2	14
9	$\frac{13}{2}$	$\frac{9}{2}$	4	$-\frac{7}{2}$	$-\frac{1}{2}$	20
9	$\frac{15}{2}$	$\frac{7}{2}$	5	$-\frac{9}{2}$	$-\frac{1}{2}$	20
$\frac{19}{2}$	6	$\frac{11}{2}$	$\frac{3}{2}$	-2	$\frac{1}{2}$	21
$\frac{19}{2}$	6	$\frac{11}{2}$	$\frac{7}{2}$	-3	$-\frac{1}{2}$	21
11	$\frac{15}{2}$	$\frac{11}{2}$	3	$-\frac{1}{2}$	$-\frac{5}{2}$	24
11	8	5	8	-6	-2	24
11	$\frac{21}{2}$	$\frac{5}{2}$	8	$-\frac{15}{2}$	$-\frac{1}{2}$	24
$\frac{23}{2}$	7	$\frac{13}{2}$	$\frac{5}{2}$	0	$-\frac{5}{2}$	25
$\frac{23}{2}$	7	$\frac{13}{2}$	$\frac{15}{2}$	-5	$-\frac{5}{2}$	25
$\frac{25}{2}$	8	$\frac{13}{2}$	$\frac{15}{2}$	-3	$-\frac{9}{2}$	27
$\frac{25}{2}$	$\frac{19}{2}$	5	$\frac{13}{2}$	$-\frac{11}{2}$	-1	27
$\frac{25}{2}$	11	$\frac{7}{2}$	$\frac{21}{2}$	-9	$-\frac{3}{2}$	27
15	$\frac{21}{2}$	$\frac{13}{2}$	5	$-\frac{9}{2}$	$-\frac{1}{2}$	32
15	11	6	8	-4	-4	32
16	9	9	3	0	-3	34
16	$\frac{21}{2}$	$\frac{15}{2}$	5	$-\frac{3}{2}$	$-\frac{7}{2}$	34
16	$\frac{21}{2}$	$\frac{15}{2}$	7	$-\frac{11}{2}$	$-\frac{3}{2}$	34
$\frac{33}{2}$	$\frac{21}{2}$	8	$\frac{7}{2}$	$-\frac{7}{2}$	0	35
$\frac{33}{2}$	$\frac{21}{2}$	8	$\frac{7}{2}$	$-\frac{1}{2}$	-3	35
$\frac{33}{2}$	11	$\frac{15}{2}$	$\frac{19}{2}$	-7	$-\frac{5}{2}$	35
$\frac{33}{2}$	13	$\frac{11}{2}$	$\frac{19}{2}$	-8	$-\frac{3}{2}$	35
18	$\frac{27}{2}$	$\frac{13}{2}$	9	$-\frac{15}{2}$	$-\frac{3}{2}$	38
$\frac{37}{2}$	11	$\frac{19}{2}$	$\frac{9}{2}$	-4	$-\frac{1}{2}$	39
$\frac{37}{2}$	$\frac{25}{2}$	8	$\frac{13}{2}$	$-\frac{11}{2}$	-1	39
21	17	6	13	-11	-2	44
$\frac{43}{2}$	12	$\frac{23}{2}$	$\frac{5}{2}$	-3	$\frac{1}{2}$	45
23	14	11	4	-4	0	48
$\frac{47}{2}$	18	$\frac{15}{2}$	$\frac{21}{2}$	-9	$-\frac{3}{2}$	49
$\frac{49}{2}$	20	$\frac{13}{2}$	$\frac{39}{2}$	-16	$-\frac{7}{2}$	51
26	$\frac{43}{2}$	$\frac{13}{2}$	17	$-\frac{29}{2}$	$-\frac{5}{2}$	54
$\frac{53}{2}$	$\frac{45}{2}$	6	$\frac{41}{2}$	$-\frac{35}{2}$	-3	55
$\frac{53}{2}$	$\frac{49}{2}$	4	$\frac{45}{2}$	$-\frac{41}{2}$	-2	55
$\frac{63}{2}$	$\frac{53}{2}$	7	$\frac{43}{2}$	$-\frac{37}{2}$	-3	65
$\frac{63}{2}$	28	$\frac{11}{2}$	$\frac{47}{2}$	-21	$-\frac{5}{2}$	65
$\frac{67}{2}$	$\frac{55}{2}$	8	$\frac{41}{2}$	$-\frac{35}{2}$	-3	69
$\frac{75}{2}$	32	$\frac{15}{2}$	$\frac{53}{2}$	-23	$-\frac{7}{2}$	77
57	$\frac{105}{2}$	$\frac{13}{2}$	52	$-\frac{95}{2}$	$-\frac{9}{2}$	116
$\frac{141}{2}$	$\frac{133}{2}$	6	$\frac{127}{2}$	$-\frac{119}{2}$	-4	143

This table has been generated for Algorithm 1 using a Fortran program on a VAX-11/780 and a Lisp program on a Symbolics computer. The values of D and E were restricted to 10.

Table 2. Polynomial zeros of degree 2 of the 6-j coefficient, using either algorithms 2 or 2a, up to a, b, c, d, e or $f \leqslant 14$. The numerator and denominator parameters of the ${}_4F_3(1)$ are also given for each entry which corrsponds to a polynomial zero of degree 2 of the 6-j coefficient. Only the first 55 entries are listed but it is a complete list up to that point, as can be verified by a comparison with table IV of Srinivasa Rao and Rajeswari (1985a).

a	b	e	d	c	f	A	B	C	D	E	F	G
6	6	3	6	5	6	−5	−3	20	−2	5	4	2
6	6	5	6	3	6	−5	−3	20	−2	5	4	2
6	6	6	6	5	3	−5	−3	20	−2	5	4	2
$\frac{13}{2}$	6	$\frac{11}{2}$	$\frac{11}{2}$	3	$\frac{11}{2}$	−5	−3	20	−2	5	4	2
7	6	4	4	6	5	−5	−3	20	−2	5	4	2
7	6	5	4	6	4	−5	−3	20	−2	5	4	2
7	$\frac{13}{2}$	$\frac{9}{2}$	4	$\frac{11}{2}$	$\frac{9}{2}$	−5	−3	20	−2	5	4	2
$\frac{15}{2}$	$\frac{11}{2}$	4	$\frac{9}{2}$	$\frac{11}{2}$	5	−5	−3	20	−2	5	4	2
$\frac{15}{2}$	$\frac{11}{2}$	5	$\frac{9}{2}$	$\frac{11}{2}$	4	−5	−3	20	−2	5	4	2
$\frac{15}{2}$	6	$\frac{9}{2}$	$\frac{9}{2}$	5	$\frac{9}{2}$	−5	−3	20	−2	5	4	2
8	8	3	$\frac{13}{2}$	$\frac{11}{2}$	$\frac{15}{2}$	−6	−3	24	−2	8	4	2
$\frac{17}{2}$	$\frac{15}{2}$	3	6	6	$\frac{15}{2}$	−6	−3	24	−2	8	4	2
$\frac{17}{2}$	$\frac{15}{2}$	6	6	3	$\frac{15}{2}$	−6	−3	24	−2	8	4	2
$\frac{17}{2}$	8	$\frac{5}{2}$	4	$\frac{7}{2}$	8	−4	−3	22	−2	11	2	1
$\frac{17}{2}$	8	$\frac{7}{2}$	4	$\frac{5}{2}$	8	−4	−3	22	−2	11	2	1
$\frac{17}{2}$	$\frac{17}{2}$	3	4	3	$\frac{15}{2}$	−4	−3	22	−2	11	2	1
9	8	3	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{15}{2}$	−4	−3	22	−2	11	2	1
9	8	4	$\frac{9}{2}$	$\frac{13}{2}$	$\frac{13}{2}$	−6	−3	24	−2	8	4	2
9	8	8	$\frac{15}{2}$	$\frac{11}{2}$	$\frac{11}{2}$	−5	−5	27	−2	6	5	5
9	$\frac{17}{2}$	$\frac{9}{2}$	$\frac{9}{2}$	6	6	−6	−3	24	−2	8	4	2
9	9	5	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{15}{2}$	−2	−2	25	−2	12	4	4
9	9	5	$\frac{15}{2}$	$\frac{15}{2}$	$\frac{7}{2}$	−10	−2	25	−2	4	4	4
9	9	8	$\frac{17}{2}$	$\frac{15}{2}$	$\frac{7}{2}$	−8	−3	28	−2	7	6	3
$\frac{19}{2}$	$\frac{15}{2}$	4	5	6	$\frac{13}{2}$	−6	−3	24	−2	8	4	2
$\frac{19}{2}$	8	$\frac{9}{2}$	5	$\frac{11}{2}$	6	−6	−3	24	−2	8	4	2
$\frac{19}{2}$	8	$\frac{15}{2}$	7	$\frac{11}{2}$	6	−5	−5	27	−2	6	5	5
$\frac{19}{2}$	$\frac{17}{2}$	8	8	8	$\frac{7}{2}$	−8	−3	28	−2	7	6	3
$\frac{19}{2}$	$\frac{19}{2}$	9	9	7	$\frac{9}{2}$	−7	−4	30	−2	8	6	4
$\frac{21}{2}$	9	$\frac{13}{2}$	6	$\frac{17}{2}$	5	−8	−3	28	−2	7	6	3
$\frac{21}{2}$	9	$\frac{17}{2}$	8	$\frac{9}{2}$	8	−7	−4	30	−2	8	6	4
$\frac{21}{2}$	$\frac{19}{2}$	6	6	8	$\frac{11}{2}$	−8	−3	28	−2	7	6	3
$\frac{21}{2}$	$\frac{19}{2}$	8	6	9	$\frac{11}{2}$	−7	−4	30	−2	8	6	4
$\frac{21}{2}$	$\frac{21}{2}$	2	5	5	$\frac{17}{2}$	−7	−2	26	−2	13	2	1
$\frac{21}{2}$	$\frac{21}{2}$	7	8	6	$\frac{13}{2}$	−7	−4	30	−2	8	6	4
11	$\frac{17}{2}$	$\frac{13}{2}$	$\frac{13}{2}$	8	5	−8	−3	28	−2	7	6	3
11	9	6	$\frac{13}{2}$	$\frac{15}{2}$	$\frac{11}{2}$	−8	−3	28	−2	7	6	3
11	$\frac{21}{2}$	$\frac{5}{2}$	$\frac{9}{2}$	5	8	−7	−2	26	−2	13	2	1
$\frac{23}{2}$	$\frac{17}{2}$	8	7	8	$\frac{11}{2}$	−7	−4	30	−2	8	6	4
$\frac{23}{2}$	$\frac{19}{2}$	7	7	7	$\frac{13}{2}$	−7	−4	30	−2	8	6	4
$\frac{23}{2}$	11	$\frac{5}{2}$	8	$\frac{17}{2}$	8	−11	−2	30	−2	10	4	2
$\frac{25}{2}$	11	$\frac{7}{2}$	7	$\frac{17}{2}$	7	−11	−2	30	−2	10	4	2
$\frac{25}{2}$	$\frac{23}{2}$	4	7	8	$\frac{13}{2}$	−11	−2	30	−2	10	4	2
$\frac{25}{2}$	12	$\frac{15}{2}$	$\frac{7}{2}$	9	$\frac{23}{2}$	−7	−4	35	−2	14	7	2
$\frac{25}{2}$	12	$\frac{17}{2}$	$\frac{5}{2}$	9	$\frac{23}{2}$	−8	−3	35	−2	14	8	1
$\frac{25}{2}$	$\frac{25}{2}$	8	$\frac{17}{2}$	$\frac{7}{2}$	11	−7	−4	35	−2	14	7	2
13	12	9	5	12	12	−8	−5	39	−2	12	9	4
13	12	12	5	12	9	−8	−5	39	−2	12	9	4

Table 2. (continued)

a	b	c	d	e	f	A	B	C	D	E	F	G
$\frac{27}{2}$	$\frac{25}{2}$	3	11	12	$\frac{17}{2}$	−15	−2	36	−2	9	6	3
$\frac{27}{2}$	$\frac{25}{2}$	5	9	12	$\frac{23}{2}$	−10	−4	39	−2	12	7	5
$\frac{27}{2}$	13	$\frac{15}{2}$	6	$\frac{23}{2}$	13	−7	−6	40	−2	14	7	5
$\frac{27}{2}$	13	$\frac{21}{2}$	$\frac{21}{2}$	5	$\frac{21}{2}$	−8	−5	39	−2	12	9	4
$\frac{27}{2}$	13	$\frac{23}{2}$	6	$\frac{15}{2}$	13	−7	−6	40	−2	14	7	5
14	12	6	5	9	10	−7	−4	35	−2	14	7	2
14	$\frac{25}{2}$	$\frac{13}{2}$	5	$\frac{17}{2}$	$\frac{19}{2}$	−7	−4	35	−2	14	7	2
14	13	7	$\frac{13}{2}$	$\frac{23}{2}$	$\frac{25}{2}$	−7	−6	40	−2	14	7	5

the polynomial zeros of degree 2 of the 3-j coefficient, for D, $E \leqslant 10$, were generated on VAX-11/780 and Symbolics computers, respectively. Common Lisp was used to write the programs based on algorithms 2 and 2a and the polynomial zeros of degree 2 of the 6-j coefficient, for E, F, $G \leqslant 14$, were generated on the Symbolics computer. Algorithm 2a was found to be much more efficient than algorithm 2.

Below the chosen limits of angular momenta, the polynomial zeros of degree 2 were found to be fewer in number than the polynomial zeros of degree 1. It is to be noted that the polynomial related to the Racah operators which occur in the study of Louck and Stein (1987) and Beyer *et al* (1986) are *not* orthogonal polynomials. The algorithms proposed here generate *all* the polynomial zeros of degree 2 of the 3-j and 6-j coefficients below the limits given for the angular momenta involved. These algorithms are simple compared to the method adopted by Louck and Stein (1987) for the 3-j coefficient and by Brudno and Louck (1985) and Beyer *et al* (1986) for the 6-j coefficient.

In the case of the 3-j coefficient, Louck and Stein (1987) obtained *all* the degree 2 (or weight 2) zeros and classified them by orbits of a discrete infinite-order subgroup of the Lorentz group SO(1, 1). In the case of the 6-j coefficient Brudno and Louck (1985) obtained the polynomial zeros of degree 2 for $\left\{\begin{smallmatrix} j_2 & j_1 & j_2 \\ 2 & j_2 & j_1\end{smallmatrix}\right\}$, $j_1 \leqslant 2j_2$ and $2j_2$ being a positive integer, as the orbit solutions of the Pell equation: $3x^2 - 4y^2 = \frac{11}{4}$, where $x \leqslant 2y - \frac{1}{2}$ and $2y$ is a positive integer $\geqslant 2$. This work was extended by Beyer *et al* (1986) who found nine cases in which (10) factors into two quadratic polynomial parts over the integers, one of which after transformation into a generalised Pell equation of the form (11), yields the polynomial zeros of degree 2. They then use the known orbit classification of solutions of Pell's equation. However, all the polynomial zeros of degree 2 cannot be obtained by this method. In this context, it is to be noted that algorithms 2 or 2a proposed here yields *all* the polynomial zeros of degree 2 of the 6-j coefficient.

We wish to point out that, using (7) and the definition of the α and β, when we write the Bargmann–Shelepin 4×3 array for the closed form expression given by Beyer *et al* (1986) for the polynomial zeros of degree 2 of the 6-j coefficient, we get

$$\begin{Bmatrix} (3U-3)/2 & (U+1)/2 & U \\ U & 2U-2 & (U+1)/2 \end{Bmatrix} = \begin{Vmatrix} 2U-2 & U & U \\ U-1 & 1 & 1 \\ U & 2 & 2 \\ 3U-4 & 2U & 2U \end{Vmatrix}$$

and

$$\begin{Bmatrix} 3U/2 & (U+1)/2 & (2U+3)/2 \\ (2U-1)/2 & 2U-1 & U/2 \end{Bmatrix} = \left\| \begin{matrix} 3U-3 & 2U-1 & 2U-1 \\ U-1 & 1 & 1 \\ U & 2 & 2 \\ 3U-1 & 2U+1 & 2U+1 \end{matrix} \right\|$$

where $U = 2, 3, \ldots$. Obviously, since the smallest entry in the Bargmann-Shelepin array defines the degree (or weight) of the polynomial zero of the 6-j coefficient, the above, viz (3.15a) and (3.15b) in Beyer *et al* (1986), are degree 1 zeros and not polynomial zeros of degree 2, as claimed by them.

The question of the physical significance of the polynomial zeros of angular momentum coefficients is still an open question and it is hoped that the detailed tables of degree 1 and degree 2 zeros, which are now available, will provide the essential background for such studies.

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